

Integrable discretizations of the Dym equation

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Abstract Integrable discretizations of the complex and real Dym equations are proposed. N -soliton solutions for both semi-discrete and fully discrete analogues of the complex and real Dym equations are also presented.

Keywords Dym equation · integrable discretization · N -soliton solution

1 Introduction

In this article, we investigate integrable discretizations of the Dym equation (often called the Harry Dym equation) [1, 2, 3, 4]

$$\begin{aligned} \text{the complex Dym equation : } r_t + r^3 r_{zzz} &= 0, \quad r, z \in \mathbb{C}, t \in \mathbb{R}, \\ \text{the real Dym equation : } r_t + r^3 r_{xxx} &= 0, \quad r, x, t \in \mathbb{R}. \end{aligned}$$

Note that the form of the complex Dym equation is same as the real Dym equation except taking complex values r and z for the complex Dym equation instead of taking real values r and x for the real Dym equation, but this appears in various physical problems [3, 4]. The Dym equation was found by Harry Dym when he was trying to transfer some results about isospectral flows to the string equation during Martin Kruskal's lectures [1, 2]. The Dym equation belongs to a class of integrable nonlinear evolution equations found by Wadati, Konno and Ichikawa [5]. It is well

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known that the Dym equation is transformed into the modified KdV (mKdV) equation by the hodograph (reciprocal) transformation [6, 7, 8, 9].

The mKdV equation

$$u_t \pm \frac{3}{2}u^2u_s + u_{sss} = 0, \quad (1)$$

where u is a real function with respect to s and t . Here the case of “+” sign corresponds to the focusing mKdV equation and the case of “−” sign corresponds to the defocusing mKdV equation. Introducing a real function $\theta(s, t)$ such that $u(s, t) = \frac{\partial}{\partial s}\theta(s, t)$, this leads to the potential mKdV equation

$$\theta_t \pm \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0. \quad (2)$$

For the focusing mKdV equation

$$u_t + \frac{3}{2}u^2u_s + u_{sss} = 0, \quad (3)$$

a reciprocal link between the potential focusing mKdV equation and the complex Dym equation is deeply related to a conservation law

$$(e^{i\theta})_t + \left(\frac{1}{2}(\theta_s)^2 e^{i\theta} + i\theta_{ss} e^{i\theta} \right)_s = 0. \quad (4)$$

Using the conserved density of this conservation law, we consider the hodograph (reciprocal) transformation [8, 9]

$$z(s, t) = \int_0^s e^{i\theta(s', t)} ds' + x_0, \quad t'(s, t) = t, \quad (5)$$

which leads to

$$\frac{\partial}{\partial s} = e^{i\theta} \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \left(-\frac{1}{2}(\theta_s)^2 e^{i\theta} - i\theta_{ss} e^{i\theta} \right) \frac{\partial}{\partial z}. \quad (6)$$

Applying (6) to (4) and introducing a new dependent complex variable $r = \frac{\partial z}{\partial s} = e^{i\theta}$, we obtain the complex Dym equation

$$r_{t'} + r^3 r_{zzz} = 0, \quad (7)$$

which leads to

$$v_{t'} + (v^{-\frac{1}{2}})_{zzz} = 0, \quad (8)$$

via $r = v^{-\frac{1}{2}}$.

Next, consider the defocusing modified KdV (mKdV) equation

$$u_t - \frac{3}{2}u^2u_s + u_{sss} = 0. \quad (9)$$

A reciprocal link between the potential defocusing mKdV equation and the real Dym equation is deeply related to a conservation law

$$(e^\theta)_t + \left(-\frac{1}{2}(\theta_s)^2 e^\theta + \theta_{ss} e^\theta \right)_s = 0. \quad (10)$$

Using the conserved density of this conservation law, we consider the hodograph (reciprocal) transformation [8, 9]

$$x(s, t) = \int_0^s e^{\theta(s', t)} ds' + x_0, \quad t'(s, t) = t, \quad (11)$$

which leads to

$$\frac{\partial}{\partial s} = e^\theta \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \left(\frac{1}{2} (\theta_s)^2 e^\theta - \theta_{ss} e^\theta \right) \frac{\partial}{\partial x}. \quad (12)$$

Applying (12) to (10) and introducing a new dependent real variable $r = \frac{\partial x}{\partial s} = e^\theta$, we obtain the real Dym equation

$$r_{t'} + r^3 r_{xxx} = 0, \quad (13)$$

which leads to

$$v_{t'} + (v^{-\frac{1}{2}})_{xxx} = 0, \quad (14)$$

via $r = v^{-\frac{1}{2}}$.

The tau-functions and bilinear equations of the complex and real Dym equations (i.e., the focusing and defocusing mKdV equations) are given as follows:

1. The complex Dym equation is transformed into the bilinear equations of the focusing mKdV equation

$$\begin{aligned} (D_s^3 + D_t) \tau \cdot \tau^* &= 0, \\ D_s^2 \tau \cdot \tau^* &= 0, \end{aligned}$$

via the dependent variable transformation

$$r = \left(\frac{\tau}{\tau^*} \right)^2, \quad \theta = \frac{2}{i} \log \frac{\tau}{\tau^*}, \quad (15)$$

and the hodograph (reciprocal) transformation

$$z(s, t) = \int_0^s e^{i\theta(s', t)} ds' + x_0, \quad t'(s, t) = t, \quad (16)$$

where τ^* is a complex conjugate of τ . In this case, there are two types of explicit soliton solutions which are N -soliton and M -breather solutions. These solutions can be expressed by Wronskians.

N -soliton solution:

$$\tau(s, t) = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tau_l^*(t) = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (17)$$

$$f_j^{(i)} = \alpha_i p_i^j e^{p_i s - p_i^3 t} + \beta_i (-p_i)^j e^{-p_i s + p_i^3 t}, \quad (18)$$

where $p_i, \alpha_i \in \mathbb{R}$, $\beta_i \in \sqrt{-1}\mathbb{R}$ for $i = 1, \dots, N$.

M -breather solution:

$$\tau(s, t) = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tau_l^*(t) = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (19)$$

$$f_j^{(i)} = \alpha_i p_i^j e^{p_i s - p_i^3 t} + \beta_i (-p_i)^j e^{-p_i s + p_i^3 t}, \quad (20)$$

where $N = 2M$, $p_i, \alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, \dots, 2M$, $p_{2k} = p_{2k-1}^*$, $\alpha_{2k} = \alpha_{2k-1}^*$, $\beta_{2k} = -\beta_{2k-1}^*$ for $k = 1, \dots, M$.

2. The real Dym equation is transformed into the bilinear equations of the defocusing mKdV equation

$$\begin{aligned} (D_s^3 + D_t)\tau \cdot \tilde{\tau} &= 0, \\ D_s^2\tau \cdot \tilde{\tau} &= 0, \end{aligned}$$

via the dependent variable transformation

$$r = \left(\frac{\tau}{\tilde{\tau}}\right)^2, \quad \theta = 2 \log \frac{\tau}{\tilde{\tau}}. \quad (21)$$

and the hodograph (reciprocal) transformation

$$x(s, t) = \int_0^s e^{\theta(s', t)} ds' + x_0, \quad t'(s, t) = t. \quad (22)$$

In this case, there is an N -cusped soliton solution which can be expressed by Wronskians:

$$\tau(s, t) = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tilde{\tau}_l(t) = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (23)$$

$$f_j^{(i)} = \alpha_i p_i^j e^{p_i s - p_i^3 t} + \beta_i (-p_i)^j e^{-p_i s + p_i^3 t}, \quad (24)$$

where $p_i, \alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, \dots, N$.

Remark 1 The above bilinear equations are obtained from the following bilinear equations which belong to the modified KP hierarchy (the two-dimensional Toda lattice hierarchy):

$$(D_s^3 + D_t + 3D_s D_{s_2})\tau(k+1) \cdot \tau(k) = 0, \quad (25)$$

$$(D_s^2 - D_{s_2})\tau(k+1) \cdot \tau(k) = 0, \quad (26)$$

by imposing the conditions

$$\frac{\partial}{\partial s_2} \tau(k) = B \tau(k), \quad \tau(k+1) = C \tau^*(k), \quad B, C \in \mathbb{R}, \quad (27)$$

(for the real Dym equation, the second condition is replaced by $\tau(k+1) = C \tilde{\tau}(k)$) and denoting $\tau = \tau(0)$.

2 Integrable semi-discrete analogues of the complex and real Dym equations

Lemma 1 *Let*

$$\tau_l(t) = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \hat{\tau}_l(t) = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (28)$$

$$f_j^{(i)} = \alpha_i p_i^j (1 - \epsilon p_i)^{-l} e^{\frac{p_i}{1 - \epsilon^2 p_i^2} t} + \beta_i (-p_i)^j (1 + \epsilon p_i)^{-l} e^{-\frac{p_i}{1 - \epsilon^2 p_i^2} t}. \quad (29)$$

These tau-functions satisfy the bilinear equations

$$D_t \tau_l \cdot \hat{\tau}_l = \frac{1}{2\epsilon} (\hat{\tau}_{l-1} \tau_{l+1} - \hat{\tau}_{l+1} \tau_{l-1}), \quad (30)$$

$$\tau_l \hat{\tau}_l = \frac{1}{2} (\hat{\tau}_{l-1} \tau_{l+1} + \hat{\tau}_{l+1} \tau_{l-1}). \quad (31)$$

Proof See [10, 11].

Theorem 1 *An integrable semi-discrete analogue of the complex Dym equation is given by*

$$\frac{dr_l}{dt} = \frac{r_l}{\epsilon} \left(\frac{r_{l+1} - r_l}{r_{l+1} + r_l} + \frac{r_l - r_{l-1}}{r_l + r_{l-1}} \right), \quad (32)$$

$$\frac{Z_{l+1} - Z_l}{\epsilon} = r_l, \quad (33)$$

where $r_l, Z_l \in \mathbb{C}$, $t, \epsilon \in \mathbb{R}$, $l \in \mathbb{Z}$. The semi-discrete complex Dym equation is transformed into the bilinear equations

$$D_t \tau_l \cdot \tau_l^* = \frac{1}{2\epsilon} (\tau_{l-1}^* \tau_{l+1} - \tau_{l+1}^* \tau_{l-1}), \quad (34)$$

$$\tau_l \tau_l^* = \frac{1}{2} (\tau_{l-1}^* \tau_{l+1} + \tau_{l+1}^* \tau_{l-1}), \quad (35)$$

via the dependent variable transformation

$$r_l = e^{i \frac{\theta_{l+1} + \theta_l}{2}} = \frac{\tau_{l+1} \tau_l}{\tau_{l+1}^* \tau_l^*}, \quad \theta_l = \frac{2}{i} \log \frac{\tau_l}{\tau_l^*}, \quad (36)$$

where τ_l^* is a complex conjugate of τ_l . The N -soliton solution is given by

$$\tau(s, t) = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tau_l^*(t) = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (37)$$

$$f_j^{(i)} = \alpha_i p_i^j (1 - \epsilon p_i)^{-l} e^{\frac{p_i}{1 - \epsilon^2 p_i^2} t} + \beta_i (-p_i)^j (1 + \epsilon p_i)^{-l} e^{-\frac{p_i}{1 - \epsilon^2 p_i^2} t}, \quad (38)$$

where $p_i, \alpha_i \in \mathbb{R}$, $\beta_i \in \sqrt{-1}\mathbb{R}$ for $i = 1, \dots, N$.

The M -breather solution is given by

$$\tau(s, t) = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tau_l^*(t) = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (39)$$

$$f_j^{(i)} = \alpha_i p_i^j (1 - \epsilon p_i)^{-l} e^{\frac{p_i}{1 - \epsilon^2 p_i^2} t} + \beta_i (-p_i)^j (1 + \epsilon p_i)^{-l} e^{-\frac{p_i}{1 - \epsilon^2 p_i^2} t}, \quad (40)$$

where $N = 2M$, $p_i, \alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, \dots, 2M$, $p_{2k} = p_{2k-1}^*$, $\alpha_{2k} = \alpha_{2k-1}^*$, $\beta_{2k} = -\beta_{2k-1}^*$ for $k = 1, \dots, M$.

Proof Here we show that the tau-functions of the bilinear equation (34) and (35) satisfy the semi-discrete Dym equation. Dividing (34) by (35), we obtain

$$\frac{d}{dt} \log \tau_l - \frac{d}{dt} \log \tau_l^* = \frac{1}{\epsilon} \frac{\tau_{l+1} \tau_{l-1}^* - \tau_{l+1}^* \tau_{l-1}}{\tau_{l+1} \tau_{l-1}^* + \tau_{l+1}^* \tau_{l-1}}. \quad (41)$$

This can be rewritten as

$$\frac{d}{dt} \log \frac{\tau_l}{\tau_l^*} = \frac{1}{\epsilon} \frac{\frac{\tau_{l+1} \tau_l}{\tau_{l+1}^* \tau_l^*} - \frac{\tau_l \tau_{l-1}}{\tau_l^* \tau_{l-1}^*}}{\frac{\tau_{l+1} \tau_l}{\tau_{l+1}^* \tau_l^*} + \frac{\tau_l \tau_{l-1}}{\tau_l^* \tau_{l-1}^*}}, \quad (42)$$

which leads to

$$\frac{d}{dt} \log \frac{\tau_l}{\tau_l^*} = \frac{1}{\epsilon} \frac{r_l - r_{l-1}}{r_l + r_{l-1}}. \quad (43)$$

Applying a shift $l \rightarrow l+1$ to (43) gives

$$\frac{d}{dt} \log \frac{\tau_{l+1}}{\tau_{l+1}^*} = \frac{1}{\epsilon} \frac{r_{l+1} - r_l}{r_{l+1} + r_l}. \quad (44)$$

Adding (43) and (44), we obtain

$$\frac{d}{dt} \log \frac{\tau_{l+1}\tau_l}{\tau_{l+1}^*\tau_l^*} = \frac{1}{\epsilon} \left(\frac{r_{l+1} - r_l}{r_{l+1} + r_l} + \frac{r_l - r_{l-1}}{r_l + r_{l-1}} \right), \quad (45)$$

which leads to (32).

Equation (33) gives the discrete hodograph transformation

$$Z_l = \sum_{j=0}^{l-1} \epsilon r_j + Z_0, \quad (46)$$

which leads to the hodograph transformation (5) in the continuous limit.

By taking care of a complex conjugacy condition of τ -functions in Lemma 1, we obtain the above constraints on parameters for soliton solutions.

Remark 2 The above semi-discrete complex Dym equation can be written in the following self-adaptive moving mesh form [12, 13]:

$$\frac{d}{dt}(Z_{l+1} - Z_l) = r_l \left(\frac{r_{l+1} - r_l}{r_{l+1} + r_l} + \frac{r_l - r_{l-1}}{r_l + r_{l-1}} \right), \quad (47)$$

$$r_l = \frac{Z_{l+1} - Z_l}{\epsilon}. \quad (48)$$

We can also obtain the following theorem about an integrable semi-discretization of the real Dym equation.

Theorem 2 *An integrable semi-discretization of the real Dym equation is given by*

$$\frac{dr_l}{dt} = \frac{r_l}{\epsilon} \left(\frac{r_{l+1} - r_l}{r_{l+1} + r_l} + \frac{r_l - r_{l-1}}{r_l + r_{l-1}} \right), \quad (49)$$

$$\frac{X_{l+1} - X_l}{\epsilon} = r_l, \quad (50)$$

where $r_l, X_l, \epsilon \in \mathbb{R}$, $l \in \mathbb{Z}$. The semi-discrete real Dym equation is transformed into

$$D_t \tau_l \cdot \tilde{\tau}_l = \frac{1}{2\epsilon} (\tilde{\tau}_{l-1}\tau_{l+1} - \tilde{\tau}_{l+1}\tau_{l-1}), \quad (51)$$

$$\tau_l \tilde{\tau}_l = \frac{1}{2} (\tilde{\tau}_{l-1}\tau_{l+1} + \tilde{\tau}_{l+1}\tau_{l-1}), \quad (52)$$

via the dependent variable transformation

$$r_l = e^{\frac{\theta_{l+1} + \theta_l}{2}} = \frac{\tau_{l+1}\tau_l}{\tilde{\tau}_{l+1}\tilde{\tau}_l}, \quad \theta_l = 2 \log \frac{\tau_l}{\tilde{\tau}_l}. \quad (53)$$

The N -cusped soliton solution is given by

$$\tau_l(t) = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tilde{\tau}_l(t) = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (54)$$

$$f_j^{(i)} = \alpha_i p_i^j (1 - \epsilon p_i)^{-l} e^{\frac{p_i}{1 - \epsilon^2 p_i^2} t} + \beta_i (-p_i)^j (1 + \epsilon p_i)^{-l} e^{-\frac{p_i}{1 - \epsilon^2 p_i^2} t}, \quad (55)$$

where $p_i, \alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, \dots, N$.

Proof The derivation of (49) from bilinear equations (51) and (52) is same as the one in Theorem 1.

Equation (50) gives the discrete hodograph transformation

$$X_l = \sum_{j=0}^{l-1} \epsilon r_j + X_0, \quad (56)$$

which leads to the hodograph transformation (11) in the continuous limit.

In the case of the semi-discrete real Dym equation, there is no constraint on parameters of soliton solutions in Lemma 1.

The above semi-discrete complex and real Dym equations arise from the motion of discrete curves [14].

3 Integrable fully discrete analogues of the complex and real Dym equations

Lemma 2 *Let*

$$\tau_n^m = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \hat{\tau}_n^m = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (57)$$

$$\begin{aligned} f_j^{(i)} &= \alpha_i p_i^j \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} \\ &\quad + \beta_i (-p_i)^j \prod_{n'}^{n-1} (1 + a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 + b_{m'} p_i)^{-1}. \end{aligned} \quad (58)$$

These tau-functions satisfy the bilinear equations

$$b_m \hat{\tau}_n^{m+1} \tau_{n+1}^m - a_n \hat{\tau}_{n+1}^m \tau_n^{m+1} + (a_n - b_m) \hat{\tau}_{n+1}^{m+1} \tau_n^m = 0, \quad (59)$$

$$b_m \tau_n^{m+1} \hat{\tau}_{n+1}^m - a_n \tau_{n+1}^m \hat{\tau}_n^{m+1} + (a_n - b_m) \tau_{n+1}^{m+1} \hat{\tau}_n^m = 0. \quad (60)$$

Proof See [10].

Theorem 3 *An integrable fully discrete complex Dym equation is given by*

$$r_n^m = Q_{n+1}^m Q_n^m, \quad (61)$$

$$\frac{Q_{n+1}^{m+1} - Q_n^m}{Q_{n+1}^{m+1} + Q_n^m} = \frac{b_m + a_n}{b_m - a_n} \frac{Q_n^{m+1} - Q_{n+1}^m}{Q_n^{m+1} + Q_{n+1}^m}, \quad (62)$$

$$Z_{n+1}^m - Z_n^m = a_n r_n^m, \quad (63)$$

where $r_n^m, Q_n^m, Z_n^m \in \mathbb{C}$, $a_n, b_m \in \mathbb{R}$, $m, n \in \mathbb{Z}$. This is transformed into

$$b_m \tau_n^{*m+1} \tau_{n+1}^m - a_n \tau_{n+1}^{*m} \tau_n^{m+1} + (a_n - b_m) \tau_{n+1}^{*m+1} \tau_n^m = 0, \quad (64)$$

$$b_m \tau_n^{m+1} \tau_{n+1}^{*m} - a_n \tau_{n+1}^m \tau_n^{*m+1} + (a_n - b_m) \tau_{n+1}^{m+1} \tau_n^{*m} = 0, \quad (65)$$

via

$$r_n^m = e^{i \frac{\theta_{n+1}^m + \theta_n^m}{2}} = \frac{\tau_{n+1}^m \tau_n^m}{\tau_{n+1}^{*m} \tau_n^{*m}}, \quad Q_n^m = \frac{\tau_n^m}{\tau_n^{*m}}, \quad \theta_n = \frac{2}{i} \log \frac{\tau_n^m}{\tau_n^{*m}}, \quad (66)$$

where τ_n^{*m} is a complex conjugate of τ_n^m . The N -soliton solution is given by

$$\tau_n^m = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tau_n^{*m} = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (67)$$

$$\begin{aligned} f_j^{(i)} = & \alpha_i p_i^j \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} \\ & + \beta_i (-p_i)^j \prod_{n'}^{n-1} (1 + a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 + b_{m'} p_i)^{-1}, \end{aligned} \quad (68)$$

where $p_i, \alpha_i \in \mathbb{R}$, $\beta_i \in \sqrt{-1}\mathbb{R}$ for $i = 1, \dots, N$.

The M -breather solution is given by

$$\tau_n^m = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tau_n^{*m} = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (69)$$

$$\begin{aligned} f_j^{(i)} = & \alpha_i p_i^j \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} \\ & + \beta_i (-p_i)^j \prod_{n'}^{n-1} (1 + a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 + b_{m'} p_i)^{-1}, \end{aligned} \quad (70)$$

where $N = 2M$, $p_i, \alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, \dots, 2M$, $p_{2k} = p_{2k-1}^*$, $\alpha_{2k} = \alpha_{2k-1}^*$, $\beta_{2k} = -\beta_{2k-1}^*$ for $k = 1, \dots, M$.

Proof We show that the tau-functions of the bilinear equation (64) and (65) satisfy the fully discrete Dym equation. Subtracting (64) from (65), we obtain

$$\tau_{n+1}^{m+1} \tau_n^{*m} - \tau_{n+1}^{*m+1} \tau_n^m = \frac{b_m + a_n}{b_m - a_n} (\tau_n^{m+1} \tau_{n+1}^{*m} - \tau_{n+1}^m \tau_n^{*m+1}). \quad (71)$$

Adding (64) and (65), we obtain

$$\tau_{n+1}^{m+1} \tau_n^{*m} + \tau_{n+1}^{*m+1} \tau_n^m = \tau_n^{m+1} \tau_{n+1}^{*m} + \tau_{n+1}^m \tau_n^{*m+1}. \quad (72)$$

Dividing (71) by (72), we obtain

$$\frac{\tau_{n+1}^{m+1} \tau_n^{*m} - \tau_{n+1}^{*m+1} \tau_n^m}{\tau_{n+1}^{m+1} \tau_n^{*m} + \tau_{n+1}^{*m+1} \tau_n^m} = \frac{b_m + a_n}{b_m - a_n} \frac{\tau_n^{m+1} \tau_{n+1}^{*m} - \tau_{n+1}^m \tau_n^{*m+1}}{\tau_n^{m+1} \tau_{n+1}^{*m} + \tau_{n+1}^m \tau_n^{*m+1}}, \quad (73)$$

which leads to

$$\frac{\frac{\tau_{n+1}^{m+1}}{\tau_{n+1}^{*m+1}} - \frac{\tau_n^m}{\tau_n^{*m}}}{\frac{\tau_{n+1}^{m+1}}{\tau_{n+1}^{*m+1}} + \frac{\tau_n^m}{\tau_n^{*m}}} = \frac{b_m + a_n}{b_m - a_n} \frac{\frac{\tau_n^{m+1}}{\tau_n^{*m+1}} - \frac{\tau_{n+1}^m}{\tau_{n+1}^{*m}}}{\frac{\tau_n^{m+1}}{\tau_n^{*m+1}} + \frac{\tau_{n+1}^m}{\tau_{n+1}^{*m}}}. \quad (74)$$

This gives (62).

Equation (63) gives the discrete hodograph transformation

$$Z_n^m = \sum_{j=0}^{n-1} a_j r_j^m + Z_0^m, \quad (75)$$

which leads to the hodograph transformation (5) in the continuous limit.

By taking care of a complex conjugacy condition of τ -functions in Lemma 2, we obtain the above constraints on parameters for soliton solutions.

We can also obtain the following theorem about an integrable fully discrete real Dym equation.

Theorem 4 *An integrable fully discrete real Dym equation is given by*

$$r_n^m = Q_{n+1}^m Q_n^m, \quad (76)$$

$$\frac{Q_{n+1}^{m+1} - Q_n^m}{Q_{n+1}^{m+1} + Q_n^m} = \frac{b_m + a_n}{b_m - a_n} \frac{Q_n^{m+1} - Q_{n+1}^m}{Q_n^{m+1} + Q_{n+1}^m}, \quad (77)$$

$$X_{n+1}^m - X_n^m = a_n r_n^m, \quad (78)$$

where $r_n^m, Q_n^m, X_n^m, a_n, b_m \in \mathbb{R}$, $m, n \in \mathbb{Z}$. This is transformed into

$$b_m \tilde{\tau}_n^{m+1} \tau_{n+1}^m - a_n \tilde{\tau}_{n+1}^m \tau_n^{m+1} + (a_n - b_m) \tilde{\tau}_{n+1}^{m+1} \tau_n^m = 0, \quad (79)$$

$$b_m \tau_n^{m+1} \tilde{\tau}_{n+1}^m - a_n \tau_{n+1}^m \tilde{\tau}_n^{m+1} + (a_n - b_m) \tau_{n+1}^{m+1} \tilde{\tau}_n^m = 0, \quad (80)$$

via

$$r_n^m = e^{\frac{\theta_{n+1}^m + \theta_n^m}{2}} = \frac{\tau_{n+1}^m \tau_n^m}{\tilde{\tau}_{n+1}^m \tilde{\tau}_n^m}, \quad Q_n^m = \frac{\tau_n^m}{\tilde{\tau}_n^m}, \quad \theta_n = 2 \log \frac{\tau_n^m}{\tilde{\tau}_n^m}. \quad (81)$$

The N -cusped soliton solution is given by

$$\tau_n^m = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tilde{\tau}_n^m = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (82)$$

$$\begin{aligned} f_j^{(i)} = & \alpha_i p_i^j \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} \\ & + \beta_i (-p_i)^j \prod_{n'}^{n-1} (1 + a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 + b_{m'} p_i)^{-1}, \end{aligned} \quad (83)$$

where $p_i, \alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, \dots, N$.

Proof The derivation of the fully discrete real Dym equation from bilinear equations (79) and (80) is same as the one in Theorem 3. In the case of the fully discrete real Dym equation, there is no constraint on parameters of soliton solutions in Lemma 2.

Remark 3 Equation (78) gives the discrete hodograph transformation

$$X_n^m = \sum_{j=0}^{n-1} a_j r_j^m + X_0^m, \quad (84)$$

which leads to the hodograph transformation (11) in the continuous limit.

The above fully discrete complex and real Dym equations arise from the motion of discrete curves [14].

Based on bilinear equations, we also obtain another form of fully discrete complex and real Dym equations.

Theorem 5 *An integrable fully discrete complex Dym equation is given by*

$$\frac{\sqrt{r_{n+1}^{m+1}} - \sqrt{r_n^m}}{\sqrt{r_{n+1}^{m+1}} + \sqrt{r_n^m}} = \frac{b_m + a_n}{b_m - a_n} \frac{\sqrt{r_n^{m+1}} - \sqrt{r_{n+1}^m}}{\sqrt{r_n^{m+1}} + \sqrt{r_{n+1}^m}}, \quad (85)$$

$$Z_{n+1}^m - Z_n^m = a_n \sqrt{r_{n+1}^m r_n^m}, \quad (86)$$

where $r_n^m, Z_n^m \in \mathbb{C}$, $a_n, b_m \in \mathbb{R}$, $m, n \in \mathbb{Z}$. This is transformed into

$$b_m \tau_n^{*m+1} \tau_{n+1}^m - a_n \tau_{n+1}^{*m} \tau_n^{m+1} + (a_n - b_m) \tau_{n+1}^{*m+1} \tau_n^m = 0, \quad (87)$$

$$b_m \tau_n^{m+1} \tau_{n+1}^{*m} - a_n \tau_{n+1}^m \tau_n^{*m+1} + (a_n - b_m) \tau_{n+1}^{m+1} \tau_n^{*m} = 0, \quad (88)$$

via

$$r_n^m = e^{i\theta_n^m} = \left(\frac{\tau_n^m}{\tau_n^{*m}} \right)^2, \quad \theta_n = \frac{2}{i} \log \frac{\tau_n^m}{\tau_n^{*m}}. \quad (89)$$

The N -soliton solution is given by

$$\tau_n^m = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tau_n^{*m} = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (90)$$

$$\begin{aligned} f_j^{(i)} &= \alpha_i p_i^j \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} \\ &\quad + \beta_i (-p_i)^j \prod_{n'}^{n-1} (1 + a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 + b_{m'} p_i)^{-1}, \end{aligned} \quad (91)$$

where $p_i, \alpha_i \in \mathbb{R}$, $\beta_i \in \sqrt{-1}\mathbb{R}$ for $i = 1, \dots, N$.

The M -breather solution is given by

$$\tau_n^m = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tau_n^{*m} = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (92)$$

$$\begin{aligned} f_j^{(i)} &= \alpha_i p_i^j \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} \\ &\quad + \beta_i (-p_i)^j \prod_{n'}^{n-1} (1 + a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 + b_{m'} p_i)^{-1}, \end{aligned} \quad (93)$$

where $N = 2M$, $p_i, \alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, \dots, 2M$, $p_{2k} = p_{2k-1}^*$, $\alpha_{2k} = \alpha_{2k-1}^*$, $\beta_{2k} = -\beta_{2k-1}^*$ for $k = 1, \dots, M$.

Proof The proof is similar to Theorem 3.

Remark 4 Equation (86) gives the discrete hodograph transformation

$$Z_n^m = \sum_{j=0}^{n-1} a_j \sqrt{r_{j+1}^m r_j^m} + Z_0^m, \quad (94)$$

which leads to the hodograph transformation (5) in the continuous limit.

Theorem 6 *The fully discrete real Dym equation is given by*

$$\frac{\sqrt{r_{n+1}^{m+1}} - \sqrt{r_n^m}}{\sqrt{r_{n+1}^{m+1}} + \sqrt{r_n^m}} = \frac{b_m + a_n}{b_m - a_n} \frac{\sqrt{r_n^{m+1}} - \sqrt{r_{n+1}^m}}{\sqrt{r_n^{m+1}} + \sqrt{r_{n+1}^m}}, \quad (95)$$

$$X_{n+1}^m - X_n^m = a_n \sqrt{r_{n+1}^m r_n^m}, \quad (96)$$

where $r_n^m, X_n^m, a_n, b_m \in \mathbb{R}$, $m, n \in \mathbb{Z}$. This is transformed into

$$b_m \tilde{\tau}_n^{m+1} \tau_{n+1}^m - a_n \tilde{\tau}_{n+1}^m \tau_n^{m+1} + (a_n - b_m) \tilde{\tau}_{n+1}^{m+1} \tau_n^m = 0, \quad (97)$$

$$b_m \tau_n^{m+1} \tilde{\tau}_{n+1}^m - a_n \tau_{n+1}^m \tilde{\tau}_n^{m+1} + (a_n - b_m) \tau_{n+1}^{m+1} \tilde{\tau}_n^m = 0, \quad (98)$$

via

$$r_n^m = e^{\theta_n^m} = \left(\frac{\tau_n^m}{\tilde{\tau}_n^m} \right)^2, \quad \theta_n = 2 \log \frac{\tau_n^m}{\tilde{\tau}_n^m}. \quad (99)$$

The N -cusped soliton solution is given by

$$\tau_n^m = \det \left(f_{j-1}^{(i)} \right)_{1 \leq i, j \leq N}, \quad \tilde{\tau}_n^m = \det \left(f_j^{(i)} \right)_{1 \leq i, j \leq N}, \quad (100)$$

$$\begin{aligned} f_j^{(i)} = & \alpha_i p_i^j \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} \\ & + \beta_i (-p_i)^j \prod_{n'}^{n-1} (1 + a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 + b_{m'} p_i)^{-1}, \end{aligned} \quad (101)$$

where $p_i, \alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, \dots, N$.

Proof The proof is similar to Theorem 4.

Remark 5 Equation (96) gives the discrete hodograph transformation

$$X_n^m = \sum_{j=0}^{n-1} a_j \sqrt{r_{j+1}^m r_j^m} + X_0^m, \quad (102)$$

which leads to the hodograph transformation (5) in the continuous limit.

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